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1978 J. Phys. A: Math. Gen. 11 2191

(http://iopscience.iop.org/0305-4470/11/11/007)

J. Phys. A: Math. Gen., Vol. 11, No. 11, 1978. Printed in Great Britain

Light ray and particle paths on a rotating disc

K McFarlane and N C McGill

Department of Theoretical Physics, University of St Andrews, St Andrews, Fife, Scotland

Received 22 March 1978, in final form 29 June 1978

Abstract. Light ray and free particle paths on a rotating disc are analysed. Previous work on this problem has been incomplete, often misleading and occasionally erroneous. It is shown how, due to the non-Euclidean nature of the spatial geometry of the disc, two apparently contradictory representations of photon trajectories on the disc are equivalent to each other. The velocity of light is calculated and shown to vary with position and with certain parameters describing the trajectory. The time of flight and distance travelled along the trajectory between two fixed reference points are shown to depend in general on the direction of flight ($A \rightarrow B$ or $B \rightarrow A$) as well as on the position of the end points A and B. Finally an examination is made of the claim by Jennison and others that a contraction of length occurs in the radial direction. It is shown that this claim is without foundation and that the results of experiments by Davies and Jennison are easily explained without resort to such a contraction.

1. Introduction

The rotating disc is one of the simplest examples of an accelerated frame of reference. It is of some interest therefore to employ the postulates and machinery of the general theory of relativity (for the special case of flat space–time) to find explicit formulae for light paths and free particle trajectories and to investigate the spatial geometry of the disc.

Previous work on this problem has been incomplete and occasionally faulty. Møller (1952) gives a brief account of the spatial geometry, obtaining but not solving the spatial geodesic equations, and discusses time dilation. Arzeliès (1966) derives solutions to the spatial geodesic equations, though not perhaps in the most convenient form, and establishes the spatial route traced out by a light ray, thereby reproducing Silberstein's (1921) early result. He also derives velocity transformation formulae, but these are restricted to velocity components in the circuital direction only.

More recently Grøn (1975) has investigated certain aspects of the spatial geometry and has calculated the velocity of light on a rotating disc. He obtains the wrong result however for the time taken for light to travel outwards from the centre of the disc to a point with radial coordinate r, and follows Møller in adopting a definition of velocity in which the infinitesimal time interval is not directly measurable by standard clocks.

Previously Jennison (1963, 1964) obtained formulae for light paths and an equation expressing apparent contraction of the disc in the radial direction. Subsequent experimental work by Davies and Jennison (1975) has been claimed to confirm the latter, and Ashworth and Jennison (1976) have recently restated the argument in favour of radial contraction. We shall show however that Jennison's formulae for light paths give only the local *direction* of a light ray correctly, and yield a misleading impression of the entire light trajectory when mapped onto a Euclidean plane, the reason being that the spatial geometry of the disc is non-Euclidean. We shall show also that there is no contraction of the disc in the radial direction, and that analysis of the Davies and Jennison experiments does not support the suggestion that such a contraction occurs.

In the following sections of this paper we shall derive equations for the paths of light rays and free particles on a rotating disc and for the velocity of each in terms of position along the trajectory. We shall be particularly concerned with respresentations of these trajectories, especially light rays, on a Euclidean plane. Other topics of interest to be discussed include the solutions to the spatial geodesic equations, the time of flight for a free particle or photon travelling between two fixed reference points, and the spatial distance along a null geodesic.

2. Coordinate and velocity transformation equations, and spatial geometry

We assume space–time is flat and consider a disc S rotating with constant angular velocity ω with respect to a frame $\bar{S}$ which is inertial everywhere. We choose plane polar coordinates $(\bar{r}, \bar{\theta})$ to identify reference points† in $\bar{S}$ lying in the same plane as the disc, the origin $\bar{r} = 0$ coinciding with the axis of rotation. If $\bar{t}$ denotes time measured in $\bar{S}$, the invariant space–time interval can be expressed as‡

$$ds^2 = d\bar{t}^2 - c^{-2}(d\bar{r}^2 + \bar{r}^2\, d\bar{\theta}^2). \tag{1}$$

Now consider the coordinate transformation

$$\bar{r} = r, \qquad \bar{\theta} = \theta + \omega t, \qquad \bar{t} = t. \tag{2}$$

In terms of the new coordinates (r, θ, t), ds^2 can be written as

$$ds^2 = dt^2(1 - \omega^2 r^2/c^2) - c^{-2}(dr^2 + r^2\, d\theta^2 + 2\omega r^2\, d\theta\, dt). \tag{3}$$

From the transformation (2), we note that the reference point (r, θ) is described in the original frame $\bar{S}$ as rotating about the origin with constant angular velocity ω. Thus the reference system which is the set of all reference points (r, θ) with constant values of r and θ may be identified with the rotating disc S.

We now apply the relativistic postulates to find the time interval $d\tau$ and length interval dl as measured on the disc using standard clocks and measuring rods respectively. The time interval is

$$d\tau = ds|_{dr=0} = dt(1 - \omega^2 r^2/c^2)^{1/2}, \tag{4}$$

where clearly $\omega r/c \leqslant 1$ for $d\tau$ to be real. This corresponds to the physical condition that no part of a real disc may have a velocity with respect to $\bar{S}$ greater than c. Equation (4) exhibits time dilation of moving clocks, as anticipated in the special theory, $d\tau$ being the time interval between the events (r, θ, t) and $(r, \theta, t + dt)$ according to a standard clock in S at the reference point (r, θ), and $dt = d\bar{t}$ being the time between these events according to clocks in the inertial frame $\bar{S}$. Alternatively equation (4) may be interpreted in terms of the effect on the rate of a clock placed in a pseudo-gravitational field (Møller 1952). We note that a clock at rest in S designed to indicate the coordinate time t would require

† See Møller (1952) for elucidation of the concept of reference point.

‡ We suppress the term involving $d\bar{z}$, since only two spatial dimensions are involved throughout this paper.

to run fast with respect to a standard clock at rest at the same reference point by the factor $(1-\omega^2r^2/c^2)^{-1/2}$.

The distance between any two neighbouring reference points in an arbitrary system of reference may be calculated either by an argument based on comparison of measuring rods in the reference system of interest and in the locally co-moving inertial frame (Møller 1952) or by an alternative argument involving a light signal passing to and fro between these reference points (Landau and Lifshitz 1971). In each case the result is the same; applied to the rotating disc we find, for the distance between reference points (r, θ) and $(r+\mathrm{d}r, \theta+\mathrm{d}\theta)$,

$$\mathrm{d}l=\left(\mathrm{d}r^2+\frac{r^2\,\mathrm{d}\theta^2}{1-\omega^2r^2/c^2}\right)^{1/2}. \tag{5}$$

The formula which yields equation (5) may be shown (Møller 1952) to be invariant under transformation of coordinates within the same system of reference, and thus contains the intrinsic spatial geometry of the reference system in question.

We note, as special cases of equation (5), that the distance between reference points (r, θ) and $(r+\mathrm{d}r, \theta)$ is

$$\mathrm{d}l_r=\mathrm{d}r, \tag{6}$$

while the distance between (r, θ) and $(r, \theta+\mathrm{d}\theta)$ is

$$\mathrm{d}l_\theta=\frac{r\,\mathrm{d}\theta}{(1-\omega^2r^2/c^2)^{1/2}}. \tag{7}$$

It follows from integration of equation (6) that the coordinate r is correctly interpreted as the distance in S between $r=0$ (the centre of the disc) and any reference point with radial coordinate r along the spatial geodesic ($\theta=$ constant) connecting them. The equation $\bar{r}=r$ therefore implies no length contraction in the radial direction, as suggested by the analysis of length contraction in the special theory and contrary to the claim made by Jennison (1964). On the other hand equation (7) implies length contraction in the rotational direction, again in accordance with the special theory (contraction only in the direction of relative motion), $r\,\mathrm{d}\theta=\bar{r}\,\mathrm{d}\bar{\theta}$ being the distance in $\bar{\mathrm{S}}$ between the reference points (r, θ) and $(r, \theta+\mathrm{d}\theta)$, their position in $\bar{\mathrm{S}}$ being determined at the same time.

From the coordinate transformation (2), we note that a span of 2π in θ is required to label the set of all events happening at the same time, since the same is true of the angle $\bar{\theta}$ in $\bar{\mathrm{S}}$. We can therefore represent the reference point (r, θ) on the disc by a point on the Euclidean plane whose polar coordinates are (r, θ). In constructing this representation we must pay careful attention to the non-Euclidean nature of the spatial geometry on the disc; thus distances in the radial direction on the Euclidean plane are correct measures of the corresponding distances on the disc, but distances in the rotational direction are underestimates of the corresponding distances on the disc by a factor $(1-\omega^2r^2/c^2)^{1/2}$. In particular, we note from equation (7) that the actual distance on the disc around one complete orbit of a circle of radius r centred on $r=0$ is $2\pi r(1-\omega^2r^2/c^2)^{-1/2}$.

To investigate the spatial geometry of the disc more fully, we require to solve the geodesic equations for the space with metric given by equation (5). Møller (1952) has discussed some aspects of these equations and has displayed a few typical spatial geodesics mapped onto a Euclidean plane. He did not quote solutions to these equations however, and we now present the solutions here†:

$$r^2 = r_0^2(1+\lambda^2), \tag{8a}$$

$$\theta = \theta_0 + \tan^{-1}\lambda - (\omega^2 r_0^2/c^2)\lambda, \tag{8b}$$

$$s = r_0(1-\omega^2 r_0^2/c^2)^{1/2}\lambda, \tag{8c}$$

where λ is a parameter, (r_0, θ_0) is the point on the arc nearest the origin, and s is the distance along the arc measured from this point. Singular solutions also exist for geodesics passing through the origin, of the form

$$r = s, \qquad \theta = \theta_0. \tag{9}$$

In figure 1(*a*) we have plotted some contour lines on the Euclidean plane joining points equidistant from the fixed point ($\omega r/c = 0{\cdot}8$, $\theta = 0$). Figure 1(*b*) shows some spatial geodesics passing through the same point.

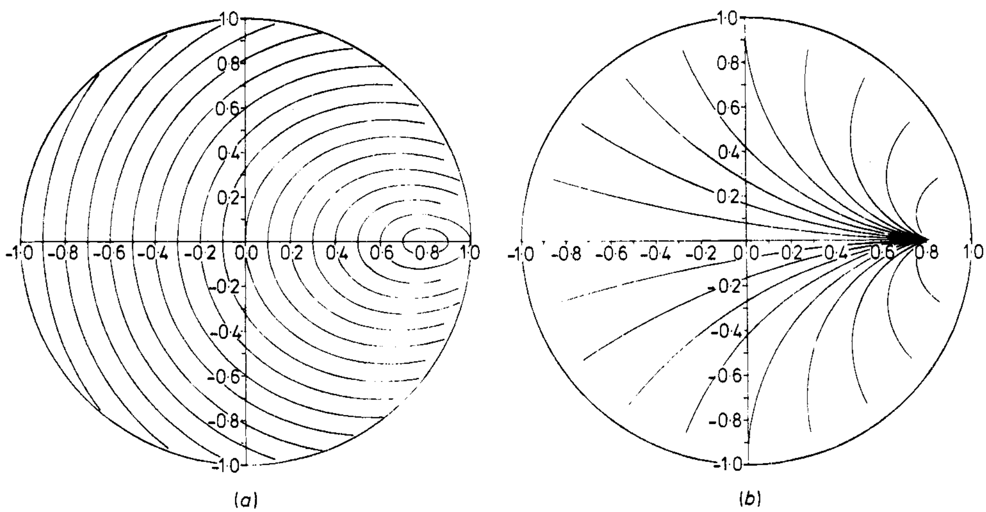

Figure 1. (*a*) Contour lines and (*b*) spatial geodesics drawn for the point ($\omega r/c = 0{\cdot}8$, $\theta = 0$). The outer circle is the 'velocity-of-light' circle with radius $R = c/\omega$. The gaps in contours and geodesic lines near this circle are due to computational difficulties and have no physical significance.

Following Landau and Lifshitz (1971) we now define the velocity of a particle which passes through the space–time events (r, θ, t) and $(r+dr, \theta+d\theta, t+dt)$ by

$$v = dl/d\tau, \tag{10}$$

dl and $d\tau$ being given by equations (5) and (4) respectively. This contrasts with the definition $v = dl/dt$ adopted by Møller (1952), Grøn (1975) and others; but we prefer to use $d\tau$ rather than dt since only the former is a time interval according to a standard clock. In addition it may easily be shown that the velocity of light is always equal to c in

† On eliminating λ from (8*a*) and (8*b*), the solution quoted by Arzeliès (1966) is recovered; the solution given here is preferred, however, since it provides additional information about arc length.

time-orthogonal frames whenever 'velocity' is defined in terms of a proper time interval, but not necessarily so when defined in terms of a coordinate time interval.

We can now derive transformation formulae for components of velocity in the radial and rotational directions. In the disc frame these components are defined by

$$v_r = \frac{\mathrm{d}l_r}{\mathrm{d}\tau} = \frac{\mathrm{d}r/\mathrm{d}t}{(1-\omega^2 r^2/c^2)^{1/2}}, \tag{11a}$$

$$v_\theta = \frac{\mathrm{d}l_\theta}{\mathrm{d}\tau} = \frac{r\,\mathrm{d}\theta/\mathrm{d}t}{1-\omega^2 r^2/c^2}. \tag{11b}$$

In the inertial frame, the corresponding components are $\bar{v}_r = \mathrm{d}\bar{r}/\mathrm{d}\bar{t}$, $\bar{v}_\theta = \bar{r}\,\mathrm{d}\bar{\theta}/\mathrm{d}\bar{t}$, and application of the transformation (2) now yields

$$v_r = \bar{v}_r/(1-\omega^2 r^2/c^2)^{1/2}, \qquad v_\theta = (\bar{v}_\theta - \omega r)/(1-\omega^2 r^2/c^2). \tag{12}$$

The total speed $v = (v_r^2 + v_\theta^2)^{1/2}$ in the rotating frame is easily shown to satisfy

$$v = [\bar{v}^2 + \omega^2 r^2(1-\bar{v}_r^2/c^2) - 2\omega r \bar{v}_\theta]^{1/2}/(1-\omega^2 r^2/c^2) \tag{13}$$

where $\bar{v}$ is the corresponding speed in the inertial frame. We note that, although $\bar{v} \leqslant c$, there is no corresponding upper limit on the values of v_r, v_θ and v. Also, the velocity transformation equations in (12) are asymmetrical between the two reference frames, as befits their different physical specification: the same results are *not* obtained by interchanging the coordinates and replacing ω by $-\omega$. In particular, $\bar{v}_r = 0$ and $\bar{v}_\theta = \omega r$ for a particle which is at rest in the frame of the disc ($v_r = v_\theta = 0$), whereas $v_r = 0$ and $v_\theta = -\omega r/(1-\omega^2 r^2/c^2)$ for a particle which is at rest in the inertial frame ($\bar{v}_r = \bar{v}_\theta = 0$).

3. Trajectory of free particles and light rays on the disc

According to relativistic theory, free particle paths in space–time are solutions of the geodesic equation, and photon paths are solutions of the null geodesic equation. These equations may be solved in the most convenient coordinate system, and the solutions in any other coordinates may be found directly from the coordinate transformation equations. For our problem, analysis of free particle and photon paths with respect to the inertial frame is elementary, and we therefore deduce, invoking the coordinate transformation (2), the following equations which describe these paths in the frame of the disc:

$$r^2 = r_0^2 + \bar{v}^2 t^2, \qquad \tan(\theta + \omega t - \theta_0) = \sigma \bar{v} t / r_0. \tag{14a}$$

Here $\bar{v}$ is the (constant) speed of the particle ($\bar{v} \geqslant 0$; $\bar{v} = c$ if the 'particle' is a photon) with respect to the inertial frame $\bar{\mathrm{S}}$, $(r_0, \theta_0, 0)$ are the event coordinates of the point on the trajectory closest to the disc centre, and $\sigma = +1$ (-1) if the rotational velocity component $\bar{v}_\theta$ in the inertial frame is positive (negative). Singular solutions also exist representing motion through the origin ($r_0 = 0$) and are of the form

$$r = \bar{v}|t|, \qquad \theta + \omega t - \theta_0 = 0, \tag{14b}$$

where θ_0 is the constant value of $\bar{\theta}$ in $\bar{\mathrm{S}}$. We note that the path traced out in this case is that of an Archimedean spiral. It can easily be shown that these solutions satisfy the appropriate geodesic equation in the (r, θ, t) coordinate system (Davies 1976), although the covariance of the theory renders this unnecessary.

The velocity components of the particle or photon can now be evaluated, either by calculating $\mathrm{d}r/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ from (14*a*) then inserting the results in (11), or by first evaluating the velocity components $\bar{v}_r$ and $\bar{v}_\theta$ in the inertial frame then applying the transformation (12). By either method we find

$$v_r = \pm\frac{\bar{v}(1-r_0^2/r^2)^{1/2}}{(1-\omega^2r^2/c^2)^{1/2}}, \qquad v_\theta = \frac{\sigma\bar{v}r_0/r - \omega r}{1-\omega^2r^2/c^2}, \tag{15}$$

$$v = \frac{c}{1-\omega^2r^2/c^2}\left[\left(1-\frac{\sigma\bar{v}}{c}\frac{\omega r_0}{c}\right)^2 - \left(1-\frac{\bar{v}^2}{c^2}\right)\left(1-\frac{\omega^2r^2}{c^2}\right)\right]^{1/2}, \tag{16}$$

where the sign of v_r is $+1$ (-1) if $t>0$ $(t<0)$.

We note that the speed of a free particle at a point on its trajectory with radial coordinate r can sometimes (for example, when $\bar{v}=0$ and $\omega(r+r_0)/c>1$) be greater than the speed of a photon passing through the same point on *its* trajectory, for the same value of r_0, provided $\sigma=1$ for the photon trajectory. No alarm need be caused by this however, since the two trajectories are quite different and the comparison has no particular significance. By contrast we shall prove in § 4 the physically expected result that the fastest signal between any two fixed reference points on the disc is a light signal.

Grøn (1975) has obtained expressions for the velocity components for a light ray on the disc which, even allowing for a difference in definition (Grøn defines $v=\mathrm{d}l/\mathrm{d}t$, in our notation, compared with our choice $v=\mathrm{d}l/\mathrm{d}\tau$), differ markedly from the expressions in (15) above. Inspection of Grøn's analysis shows however that his expressions for v_r and v_θ are obtained by setting $\mathrm{d}s=0$ in equation (3) and solving for $\mathrm{d}r/\mathrm{d}t$ or $\mathrm{d}\theta/\mathrm{d}t$ with $\mathrm{d}\theta$ or $\mathrm{d}r$ respectively set equal to zero. Thus his calculated velocity components are correct only if the light ray is momentarily radial or circuital in direction and do not apply at a general point on the light ray. To verify this, we note from (15) that $v_\theta=0$ if and only if $\sigma=1$ and (setting $\bar{v}=c$ for a light ray) $r_0/r=\omega r/c$. Substituting we find $v_r=\pm c$ which is Grøn's result (equation (47) in his paper) after allowing for the difference in definition of velocity. Similarly $v_r=0$ in (15) if and only if $r=r_0$; this yields $v_\theta=(\sigma c-\omega r_0)/(1-\omega^2r_0^2/c^2)$ which is essentially Grøn's result (his equation (50)). It is clear that, because his expressions for v_r and v_θ are valid only at special points on any particular light path, wrong results will be obtained on attempted integration of them along a finite section of the path. For example, Grøn attempts to calculate the coordinate time interval required for a light signal to travel from the origin to a point with radial coordinate r by integrating his equation for the radial component of velocity. The answer obtained (his equation (56)) is inevitably wrong, the correct result being simply r/c as can be seen from (14*b*) with $\bar{v}=c$.

Grøn's results for v_r and v_θ have been criticised by Davies (1976), who maintains that $\mathrm{d}t$ is a time interval for an observer at the origin in contrast to $\mathrm{d}\tau$ which is a local time interval; 'velocity' must therefore be $\mathrm{d}l/\mathrm{d}\tau$ and not $\mathrm{d}l/\mathrm{d}t$. We do not accept however Davies' premise that the coordinates (r, θ, t) which label events are somehow associated with one particular observer (the observer at the origin). In particular we regard $\mathrm{d}t$ as possessing a perfectly objective, observer-free meaning: in this context, it is the difference in time coordinate between the two events whose coordinates in the disc frame are (r, θ, t) and $((r+\mathrm{d}r, \theta+\mathrm{d}\theta, t+\mathrm{d}t)$. Certainly it is possible to argue—as we do here—that Grøn's choice of velocity definition $(v=\mathrm{d}l/\mathrm{d}t)$ is less preferable than an alternative $(v=\mathrm{d}l/\mathrm{d}\tau)$; but it is quite unjustified to maintain that Grøn's results are simply mistaken. As we have shown above, Grøn's equations for v_r and v_θ have restricted validity, and the only problem is that they can be so readily misinterpreted as applying along an entire light trajectory rather than at special points on it.

We now define the local direction of motion of the particle or photon as the solution ϕ to the equations $v_r = v \cos \phi$, $v_\theta = v \sin \phi$. In particular, for a photon we deduce from equations (15) and (16) that

$$\cos \phi = \pm \frac{(1 - r_0^2/r^2)^{1/2}(1 - \omega^2 r^2/c^2)^{1/2}}{1 - \sigma\omega r_0/c}, \qquad \sin \phi = \frac{\sigma r_0/r - \omega r/c}{1 - \sigma\omega r_0/c}. \tag{17}$$

Solving the $\sin \phi$ equation for $\sigma\omega r_0/c$ and substituting in equation (16) with $\bar{v}/c = 1$, we find

$$v = \frac{c}{1 + (\omega r/c)\sin \phi}. \tag{18}$$

Thus the speed of light equals c only if $r = 0$ or if $\sin \phi = 0$, i.e. $\sigma = 1$ and $r_0/r = \omega r/c$.

It is possible, and for some purposes convenient, to construct a representation of the light path on a Euclidean plane in which the relation between ϕ and r on the disc is correctly exhibited; thus ϕ will be the angle between the tangent at a point on the light path and the straight line joining this point to the origin. In addition we are free to choose *one* point on the trajectory whose coordinates are correctly represented; let this point be (r_0, θ_0). It is then easy to show that, in such a representation, the (r, ϕ) relation in (17) describes a circle of radius $\frac{1}{2}(R - \sigma r_0)$ with centre at $(\frac{1}{2}(R + \sigma r_0), \theta_0 + \frac{1}{2}\pi(1-\sigma))$ which is tangential to the 'velocity-of-light circle' with radius $R = c/\omega$. Figure 2 shows sample representative circles for $\sigma = +1$ and -1. For each, $\phi = 3\pi/2$ when $r = R$ and $\phi = \pi/2$ $(\sigma = 1)$ or $3\pi/2$ $(\sigma = -1)$ when $r = r_0$.

It is important to note that, due to the non-Euclidean nature of the spatial geometry on the disc, such a diagram is *not* the same as that which correctly portrays the (r, θ) relation. In mapping the trajectory of a light ray on the disc onto a Euclidean plane, one

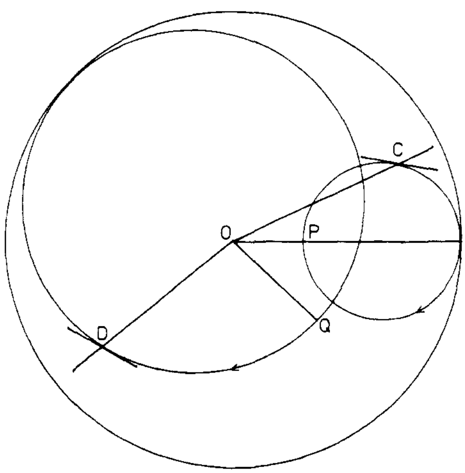

Figure 2. Representation of equation (17). The outer circle has radius R. The inner circles indicate the instantaneous direction ϕ of the light ray for $\sigma = 1$ (smaller circle) and $\sigma = -1$ (larger circle). For the smaller circle, the angle ϕ at C is the angle between OC and the tangent to the circle at C, $r = \text{OC}$, $r_0 = \text{OP}$, $\theta_0 = 0$ by choice, and l is the distance along the arc PC. For the larger circle, similar remarks apply with C and P replaced by D and Q respectively, except that θ_0 is now the exterior angle XOQ.

may freely choose between: (*a*) a diagram in which the reference point (r, θ) on the disc is represented by the point (r, θ) on the Euclidean plane (as mentioned in § 2) but in which angles and directions (and distances, in general) are not correctly represented; and (*b*) a diagram which represents correctly directions (and distances†) along the trajectory, but not the trajectory itself in the sense of the (r, θ) relation. Obviously figure 2 is a diagram of type (*b*); by contrast we exhibit in figure 3 a diagram of type (*a*) which shows the spiral paths traced out by various null geodesics starting from the fixed point $(\omega r/c = 0{\cdot}5, \theta = 0)$.

It is now clear that the circular trajectories displayed by Jennison (1963) and by Ashworth and Jennison (1976) are merely special cases of representational diagrams of type (*b*). However the two types of diagram are not distinguished in these papers, and the suggestion that such a diagram portrays the 'real' trajectory (with the implication that the diagram of type (*a*) is somehow incorrect) is certainly unjustified. We note also that the distinction between the two types of diagram has little or nothing to do with imagined experimental procedures for measuring variables of interest, contrary to the views expressed by Davies and Ashworth (1977) in a reply to Browne (1977).

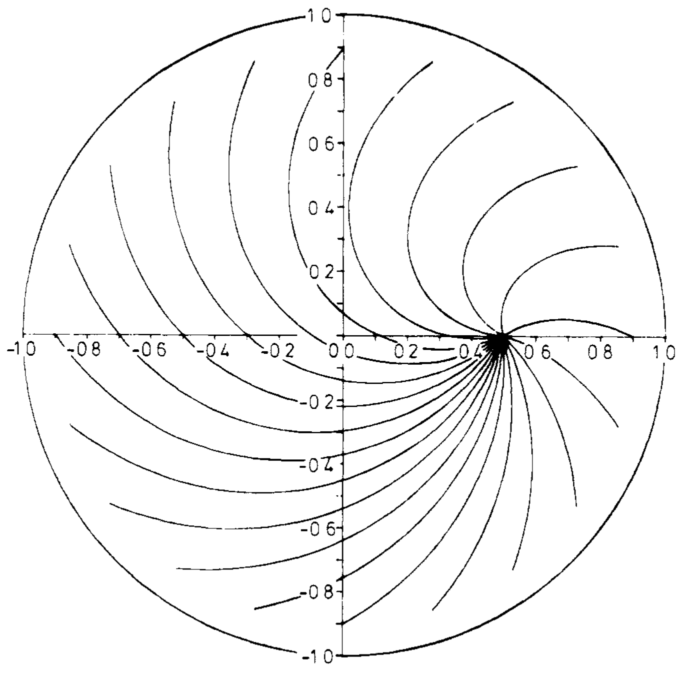

Figure 3. Paths traced out (in the sense of correct portrayal of the (r, θ) relationship) by null geodesics departing from $(\omega r/c = 0{\cdot}5, \theta = 0)$. Incoming null geodesics would be represented by a similar spiral pattern but in the opposite rotational sense.

4. Time of flight between two fixed reference points on the disc

Consider a free particle which, in the inertial frame $\bar{S}$, passes between the space–time events $(\bar{r}_1, \bar{\theta}_1, \bar{t}_1)$ and $(\bar{r}_2, \bar{\theta}_2, \bar{t}_2)$, its velocity with respect to $\bar{S}$ being $\bar{v}$. From inspection

† We shall prove this in § 5.

of the triangle in $\bar{S}$ whose vertices are $(\bar{r}_1, \bar{\theta}_1)$, $(\bar{r}_2, \bar{\theta}_2)$ and the origin (figure 4), we see immediately that

$$\bar{v}^2(\bar{t}_2-\bar{t}_1)^2=\bar{r}_1^2+\bar{r}_2^2-2\bar{r}_1\,\bar{r}_2\cos(\bar{\theta}_2-\bar{\theta}_1). \tag{19}$$

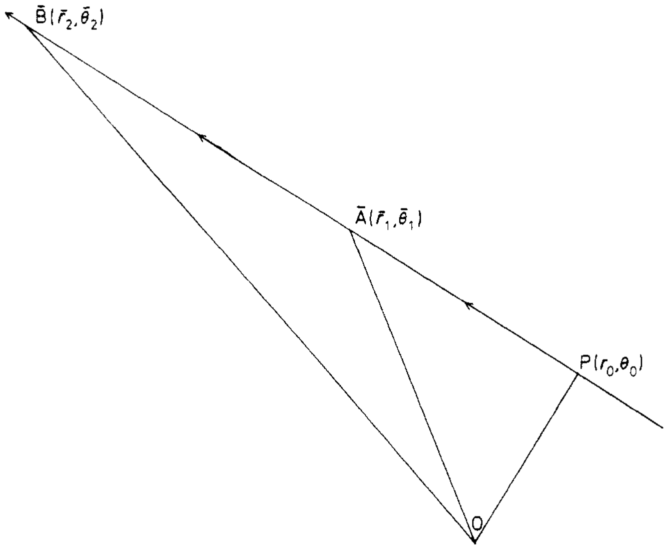

Figure 4. Spatial path traced out by particle (or photon) in the inertial frame $\bar{S}$. $\bar{A}$ and $\bar{B}$ are the points in $\bar{S}$ marking the intersection of the particle path with the world lines of the reference points $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ respectively. P is the point on the trajectory nearest the origin.

We deduce from the coordinate transformation (2) that

$$\bar{v}^2(t_2-t_1)^2=r_1^2+r_2^2-2r_1\,r_2\cos[\theta_2-\theta_1+\omega(t_2-t_1)]. \tag{20}$$

This equation therefore yields the coordinate time interval t_2-t_1 for the particle to travel from the reference point (r_1, θ_1) to the reference point (r_2, θ_2) on the disc.

An exact analytic solution of equation (20) in closed form is not possible, but we can nevertheless shed some light on its properties. We draw attention first to some special solutions. When $r_1=0$ we have, as expected, $t_2-t_1=r_2/\bar{v}$; correspondingly when $r_2=0$, $t_2-t_1=r_1/\bar{v}$. When $\bar{v}=0$, a solution exists provided $r_1=r_2$, namely $\omega(t_2-t_1)=\theta_1-\theta_2+2n\pi$ ($n=$ integer). This corresponds of course to the case of a particle at rest in the inertial frame which periodically coincides with the position of any reference point on the disc having the same value of r, the coordinate time interval between successive coincidences being $2\pi/\omega$.

Next we note that interchange of the coordinates (r_1, θ_1) and (r_2, θ_2) does not leave equation (20) unchanged. Hence, in general, the time required to pass between two reference points A and B, for a fixed value of $\bar{v}$, depends not only on the positions of A and B but also on whether the path is $A\to B$ or $B\to A$. Furthermore, although $\bar{v}$ in (20) is obviously a single-valued function of t_2-t_1, assuming the end points are fixed, we shall see later that t_2-t_1 is in general a multi-valued function of $\bar{v}$. Physically this arises because, provided $\bar{v}$ is small enough, a free particle with speed $\bar{v}$ relative to $\bar{S}$ starting out from (r_1, θ_1) at coordinate time t_1 can intersect the point (r_2, θ_2) at a later time by setting out in more than one direction. (The possibility of doing so can be most easily appreciated by considering the motion in the inertial frame.)

We now show that $t_2 - t_1$ takes on its minimum possible value for a light ray. We begin by expressing equation (20) in the non-dimensional form

$$\beta\eta^2 = x_1^2 + x_2^2 - 2x_1 x_2 \cos(\theta_2 - \theta_1 + \eta) \tag{21}$$

where

$$\beta = \bar{v}^2/c^2, \qquad x_1 = \omega r_1/c, \qquad x_2 = \omega r_2/c, \qquad \eta = \omega(t_2 - t_1). \tag{22}$$

The condition for $\beta(\eta)$ to have a stationary value is $\mathrm{d}\beta/\mathrm{d}\eta = 0$. Combining this condition with equation (21) we find

$$\beta^2\eta^4 - [2\beta(x_1^2 + x_2^2) - 4\beta^2]\eta^2 + (x_1^2 - x_2^2)^2 = 0. \tag{23}$$

It is convenient now to distinguish two cases: (*a*) $x_1 = x_2 \neq 0$; in this case a real solution exists for η (excluding the trivial solution $\eta = 0$) if and only if $\beta < x_1^2$; (*b*) $x_1 \neq x_2$; in this case a solution exists for η if and only if $\beta \leq \min(x_1^2, x_2^2)$. Hence, noting that $x_1 < 1$, $x_2 < 1$, in accordance with the condition that no part of a real disc can have a velocity greater than c with respect to the inertial frame $\bar{\mathrm{S}}$, we deduce that, in each case, stationary values do occur but only for $\beta < 1$.

We now observe that the right-hand side of equation (21) has, for fixed values of x_1 and x_2, a maximum possible value of $(x_1 + x_2)^2$ and a minimum possible value of $(x_1 - x_2)^2$. Thus

$$(x_1 - x_2)^2/\eta^2 \leq \beta \leq (x_1 + x_2)^2/\eta^2, \tag{24}$$

so β increases without limit as $\eta \to 0$. But β has stationary values only when $\beta < 1$. Hence a pair of values ($\beta = \beta_0 < 1$, $\eta = \eta_0$) must exist which satisfy equation (21) and for which $\beta(\eta)$ is a monotonic increasing function of η as η decreases towards 0 starting from η_0. Eventually, as η decreases, β will reach the value 1, which is the maximum value of β permitted on *physical* grounds, and must therefore correspond to the minimum allowed value of η. Clearly this value of η—the flight time for a photon—is unique.

Figure 5 exhibits the (β, η) relationship for $x_1 = 0{\cdot}8$, $\theta_1 = 0$, and $x_2 = 0{\cdot}5$, $\theta_2 = \pi$, and illustrates the various remarks made in the previous two paragraphs. As $\theta_2 - \theta_1$ is varied, the curve representing $\beta(\eta)$ alters, most noticeably in respect of the 'phase' of the oscillations which occur for small values of β, but the upper and lower bounds specified in (24) remain constant.

5. Spatial distance along null geodesics

The spatial distance between two reference points on the disc along a null geodesic joining them is not the same as the distance between them along the spatial geodesic joining them, since the two spatial trajectories are different. The latter has already been given in equations (8*a*, *b*, *c*). To calculate the former, we merely integrate $\mathrm{d}l$ given by equation (5) along the null geodesic (14*a*) with $\bar{v} = c$. The result is

$$l(r) = (c/\omega)(1 - \sigma\omega r_0/c)\sin^{-1}[(\omega/c)(r^2 - r_0^2)^{1/2}(1 - \omega^2 r_0^2/c^2)^{-1/2}], \tag{25}$$

where $l(r)$ is the distance between either of the two reference points on the light path with r as radial coordinate and the reference point on the path nearest to the origin. This

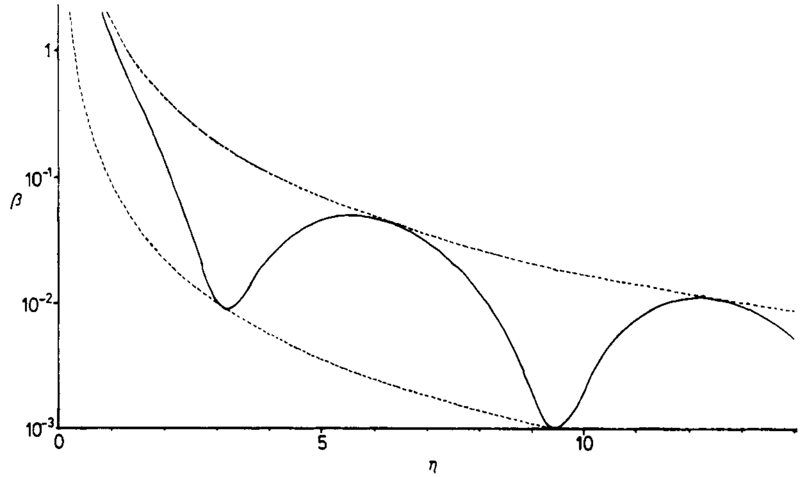

Figure 5. Relation between β and η satisfying equation (21) with $x_1 = 0{\cdot}8$, $\theta_1 = 0$ and $x_2 = 0{\cdot}5$, $\theta_2 = \pi$.

is the same result as obtained by Ashworth and Jennison (1976), although the analysis carried out by these authors involved a complicated sequence of imagined radar measurements of infinitesimal distances along the null geodesic.

A simple geometrical calculation shows that l is correctly represented by the corresponding distance along the appropriate circular arc in a diagram of the type shown in figure 2. Thus such a diagram portrays the local direction of motion *and* distance travelled, along the path of the null geodesic.

It is of interest to examine whether or not the distance travelled by a photon going from A to B is the same as the distance travelled by a photon going from B to A, A and B being the reference points (r_1, θ_1) and (r_2, θ_2) respectively. The required distance for the route A → B is

$$l_{\mathrm{AB}} = \begin{cases} \text{(i)} & l(r_2) + l(r_1) \\ \text{or (ii)} & |l(r_2) - l(r_1)| \end{cases} \tag{26}$$

where (i) applies if the route A → B includes the point on the trajectory closest to the origin and (ii) applies if it does not. In order to compare l_{AB} with l_{BA} (and indeed to make equation (25) more useful) we require to show how the end points A and B and the route (here taken to be A → B) determine the values of r_0 and σ and the choice of (i) or (ii) in equation (26).

Considering once again the trajectory of the photon in the inertial frame $\bar{\mathrm{S}}$ (figure 4), a simple trigonometrical argument shows that

$$r_0 = \frac{\bar{r}_1 \bar{r}_2 \sin(\bar{\theta}_2 - \bar{\theta}_1)}{c(\bar{t}_2 - \bar{t}_1)}. \tag{27}$$

The figure however shows a trajectory for which $\sigma = 1$, i.e. $\bar{\theta}_2 > \bar{\theta}_1$; for the other case where $\sigma = -1$ the result for r_0 is the same except that $\bar{\theta}_2 - \bar{\theta}_1$ is replaced by $\bar{\theta}_1 - \bar{\theta}_2$. Thus, making the transformation to the disc coordinates (r, θ, t), the general result is

$$r_0 = \frac{r_1 r_2 \sin|\theta_2 - \theta_1 + \omega(t_2 - t_1)|}{c(t_2 - t_1)}, \tag{28}$$

where $t_2 - t_1$ is obtained from equation (20) with $\bar{v} = c$. The null geodesic solution $r^2 = r_0^2 + c^2 t^2$ may now be used to give $|t_1|$ and $|t_2|$. Since t_2 (time of arrival at B) is always greater than t_1 (time of departure from A) we see that if $r_2 > r_1$, t_2 is positive; if $r_2 < r_1$, t_1 is negative; and if $r_1 = r_2$ (excluding the trivial case where $\theta_1 = \theta_2$) t_2 is positive and t_1 is negative. Since $t_2 - t_1$ is presumed known by this stage, the individual values of t_1 and t_2 can now be found. Then if t_1 and t_2 are of opposite sign, we take option (i) in equation (26); otherwise option (ii). We can now calculate $\bar{\theta}_1 = \theta_1 + \omega t_1$, $\bar{\theta}_2 = \theta_2 + \omega t_2$. If $\bar{\theta}_2 > \bar{\theta}_1$, $\sigma = 1$; if $\bar{\theta}_2 < \bar{\theta}_1$, $\sigma = -1$, and if $\bar{\theta}_1 = \bar{\theta}_2$ the null geodesic passes through the origin. Finally, if required, θ_0 can be calculated from the null geodesic solution for θ (equation (14*a*), setting $\bar{v} = c$) inserting the values of θ and t appropriate to the reference points A or B.

Since an exact analytic solution of equation (20) does not appear to be possible, we cannot provide explicit formulae for r_0, θ_0, σ, t_1 and t_2 in terms of the coordinates of A and B. Nevertheless, having noted in § 4 that $t_2 - t_1$ is, in general, not the same for A → B as for B → A, it is sufficiently clear that l_{BA} will not, in general, be the same as l_{AB}. There is nothing particularly surprising about this, for the two spatial trajectories that we are considering are always distinct.

There is one case of special interest however where $l_{AB} = l_{BA}$. This is where the point A is the origin O, so that $r_1 = 0$. Obviously $r_0 = 0$ for both outgoing and incoming light paths (O → B and B → O). We see directly from equation (25) that

$$l_{OB} = l_{BO} = (c/\omega)\sin^{-1}(\omega r_2/c). \tag{29}$$

As expected, this is always greater (except when $r_2 = 0$) than the shortest possible distance between O and B, which is r_2.

6. Contraction in the radial direction?

In disagreement with the results set out in § 2, Jennison (1964) has argued that a contraction of length occurs in the radial direction in the disc frame. This suggestion is based on analysis of a hypothetical experiment involving repeated to-and-fro light signals passing between an observer O at the origin and an observer B at rest on the disc at a reference point whose radial coordinate is r. The ratio of the frequencies of successive pulses recorded by the two observers is obtained from the general formula

$$\frac{\nu_2}{\nu_1} = \frac{(1 - \omega^2 r_1^2/c^2)^{1/2}}{(1 - \omega^2 r_2^2/c^2)^{1/2}} \tag{30}$$

where the source and receiver are at rest at r_1 and r_2 respectively. This formula may be proved in a number of ways (Lee and Ma 1962, Synge 1963); applied to the problem under consideration we find

$$\frac{\nu_B}{\nu_O} = (1 - \omega^2 r^2/c^2)^{-1/2}. \tag{31}$$

Jennison now asserts that according to observer B the distance r' between the two observers must therefore be

$$r' = r(1 - \omega^2 r^2/c^2)^{1/2}. \tag{32}$$

† This proposition is true, in general, only for *infinitesimally* separated end points, as the argument in Landau and Lifshitz (1971) makes clear.

It is difficult to know, however, precisely what this distance is supposed to represent, since Jennison specifies only the end points (O and B) and not the path between them. One possibility is that, for a light path, measurement of a finite to-and-fro travel time is tacitly taken to be equivalent to measurement of the to-and-fro distance (along the light path) between the end points†, the ratio of these quantities being always equal to c; yet we have seen in § 5 that the distance between O and B along either of the light paths joining them (i.e. O → B or B → O) is $(c/\omega)\sin^{-1}(\omega r/c)$ which is not the same as r', nor is it the to-and-fro travel time multiplied by a constant factor. (From equations (4) and (14b), the to-and-fro travel time is $2r/c$ according to a coordinate clock or standard clock at O, and $2r/c$ or $(2r/c)(1-\omega^2r^2/c^2)^{1/2}$ according to a coordinate clock or standard clock respectively at B.) On the other hand, neither is r' equal to the minimum distance between O and B, i.e. the distance along the spatial geodesic joining them, since equation (9) shows that this distance is equal to r. Presumably equation (32) is not merely intended to be a definition of r', since it would then be logically impossible to 'confirm' the relation experimentally, which is assumed to be possible by Davies and Jennison (1975). If, as seems to us, the distance between two arbitrary reference points is effectively *defined* by Jennison to be always $c/2$ times the to-and-fro travel time as measured by a standard clock, then we would comment: (a) that this definition is quite different from, and in conflict with, the well established and accepted concept of distance within an arbitrary system of reference as formulated by Møller (1952), Landau and Lifshitz (1971) and others; (b) that, this being so, confusion is likely to result from its use; and (c) that equation (32) is then by definition equivalent to equation (31) with exactly the same physical content, and is therefore superfluous.

We now consider briefly the experiments of Davies and Jennison (1975) whose results, the authors claim, corroborate equation (32). (We shall consider here only their first experiment, since their second experiment does not involve light signals passing in the radial direction, and in any case the authors appear to draw no theoretical conclusions from it.) Davies and Jennison demonstrated that the frequency of light pulses on arrival back at the origin after a to-and-fro journey to a fixed mirror on the periphery of a rotating disc is the same as the original frequency of pulse emission; experimentally, no appreciable shift was observed in the interference fringe pattern as the angular frequency of rotation of the disc was varied. But this is precisely to be expected on the foregoing theory, as can be seen from: (a) that two successive applications of the frequency shift formula (30) yield the expected zero shift in frequency at the end of the round trip; and (b) that the optical path length for the return trip expressed in local wavelengths, i.e. the value of $\oint dl/\lambda$, is the same in the disc frame as it is in the inertial frame, namely $2r/\lambda_0$, and is therefore independent of ω. Hence there is no need to invoke equation (32) in order to explain the experimental null result. Indeed Davies and Jennison concede that a number of assumptions are required for the validity of equation (32), two of which have been shown here to be inadmissible for the rotating reference frame.

7. Conclusions

We established in § 2 that the coordinate system (r, θ, t), related to the inertial coordinates $(\bar{r}, \bar{\theta}, \bar{t})$ by equation (2), corresponds to the rotating disc as reference system. A particular selection of spatial coordinates (r, θ) identifies a fixed reference point on the disc, while t is the time coordinate of an event according to a coordinate clock at the appropriate reference point. The relation between the rate of a coordinate clock and a standard clock placed at the same reference point is shown by equation (4).

The distance between two neighbouring reference points on the disc was given in equation (5). We noted that the spatial geometry is non-Euclidean, and gave in equations (8) and (9) the solutions to the spatial geodesic equation, i.e. the equations representing paths of minimum distance between reference points.

We then considered the motion of free particles and photons on the disc, which are obtained as solutions to the space–time geodesic and null geodesic equations. One of the problems encountered was how to represent the trajectories involved, given that the spatial geometry of the disc is non-Euclidean. We discussed, for a light ray, two useful representations of trajectories involving the Euclidean plane: one of these preserves the (r, θ) relation but gives a misleading impression of direction of motion and distance travelled, while the other achieves the opposite. No question is involved of only one of these being 'correct', contrary to the impression which may have been created by previously published work.

From the solutions to the geodesic and null geodesic equations, most other quantities of interest can be calculated, e.g. the velocity of free particles and photons at any point on their trajectories. We emphasise that velocity is a defined quantity and more than one definition is possible. We prefer $v = \mathrm{d}l/\mathrm{d}\tau$ rather than $v = \mathrm{d}l/\mathrm{d}t$; but whichever of these is adopted, the velocity of light relative to the disc is not generally equal to c. This is because the disc frame is not a time-orthogonal frame of reference.

We showed in § 4 and § 5 that the time of flight, and distance travelled, for a photon passing between two reference points (A and B) on the disc depend not only on the positions of A and B but also on whether the trajectory is A→B or B→A. This is because the spatial route traced out on the disc is quite different in the two cases, unlike the corresponding situation in an inertial frame, and conflicts in no way with the axiomatic requirement (guaranteed here by the form of equation (5)) that the distance between A and B is the same as the distance between B and A measured along the *same* spatial route.

Finally, we have argued against the suggestion made by Jennison and others that a contraction of length in the radial direction occurs on the disc, and have shown how the experimental results of Davies and Jennison (1975) are easily explained on conventional theory. The error in this suggestion consists in: (i) the faulty interpretation of the coordinates (r, θ, t) as pertaining somehow to one particular observer—the one at the origin—instead of applying to the disc frame as a whole; and (ii) an attempted 'operational definition' of distance which is inconsistent with accepted ideas of distance between neighbouring reference points within arbitrary reference systems. On the other hand a contraction of length does occur in the circuital direction, as indicated by equation (7) and in accordance with the elementary analysis of this phenomenon in special relativity.

Acknowledgments

One of us (KM) acknowledges the financial support of an SRC Research Studentship.

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